March 30: Krull Domains and the Mori-Nagata Theorem, part 2

Maijevic's Theorem

One of the main tools we need in the proof of the Mori-Nagata theorem is the theorem of J. Matijevic on the global transform of a Noetherian ring. Matijevic's theorem is a generalization of the Krul-Akizuki theorem.

Definition. Let R be a Noetherian ring with total quotient ring K. The *global transform of R* is the set T of elements $x \in K$ such that $(R : x)$ has the property that $R/(R : x)$ is Artinian, i.e., $R/(R : x)$ is zero-dimensional.

Equivalently, T consists of the set of elements $x \in K$ such that $(R : x)$ contains a product of maximal ideals.

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Remarks. 1. It is easy to check that T is a subring of K containing R.

In fact, if $x, y \in T$, $Jx \subseteq R$, $Iy \subseteq R$, and each I, J is a product of (possibly different) maximal ideals, then JI is a product of maximal ideals and $JIxy \subseteq R$. Thus, $xy \in T$. The proof that $x + y \in T$ is similar.

2. If R has dimension one, then K is the global transform of R. Indeed, if $x = \frac{a}{b} \in K$, then $b \in (R : x)$. Since b is a non-zerodivisor, $\dim(R/bR) = 0$. Thus, $\dim(R/(R : x)) = 0$.

3. Suppose $(R, \mathfrak{m}, k)$ is a local ring. Then the global transform is the set of elements $x \in K$ such that there exists $n \geq 1$ with $\mathfrak{m}^n \cdot x \subseteq R$. This is the so-called *ideal transform* of $\mathfrak{m}$.

4. In fact, for any idea $I \subseteq R$, the ideal transform of I is defined to be the set of elements $x \in K$ such that $I^n x \subseteq R$, for some n.

Ideal transforms played a central role in Nagata's construction of a counterexample to Hilbert's 14th problem.

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Theorem D2. (Matijevic) Let R be a Noetherian ring with total quotient ring K and write T for the global transform of R.

Then for any ring $R \subseteq A \subseteq T$ and non-zerodivisor $x \in R$, A/xA is a finite R-module. In particular, A/xA is a Noetherian ring.

Proof. The second statement follows immediately from the first statement.

For the first statement, it suffices to show that $A \subseteq Rx^{-n} + xA$, for some $n \geq 1$. For then, as R-submodules of K, we have

$$A/xA \subseteq (Rx^{-n} + xA)/xA \cong Rx^{-n}/(Rx^{-n} \cap xA),$$

so that A/xA is a submodule of a cyclic R-module.

Fix $a \in A$.

We first show there exists $k \geq 1$ such that $a \in Rx^{-k} + xA$.

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Let $J := (R : a)$, so that R/J is Artinian. Then the images of the ideals (x^n) in R/J form a descending sequence, which ultimately stabilizes.

Thus, there exists $k \geq 1$ such that $(x^k) + J = (x^{k+1}) + J$. Write $x^k = rx^{k+1} + j$, for some $r \in R$, $j \in J$.

Multiplying by a gives $ax^k = arx^{k+1} + aj$, and hence $a = arx + ajx^{-k}$. But $aj \in R$, so $a \in Ax + Rx^{-k}$, as required.

Now consider the R-module $(xA \cap R)/xR$. On the one hand, it is generated by the images in R/xR of finitely many elements of the form $a_i x$. On the other hand, there is a product J of maximal ideal such that $Ja_i \in R$, for all i.

Thus $J(a_i x) \subseteq xR$, for all i, so J annihilate $(xA \cap R)/xR$.

Thus, $(xA \cap R)/xR$ is a finitely generated module annihilated by a zero-dimensional ideal, and must therefore have finite length, i.e., $(xA \cap R)/Rx$ is Artinian.

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Therefore, the descending sequence of submodules $(x^h A \cap R, Rx)/Rx$ stabilizes.

In R, the descending sequence of ideals $I_h = (x^h A \cap R, xR)$ stabilizes, at say, $h = n$. We claim $A \subseteq Rx^{-n} + xA$.

Suppose $a \in A$ does not belong to $x^{-n}R + xA$. There exists k with $a \in Rx^{-k} + xA$. Note $k > n$.

Choose k minimal and write $a = rx^{-k} + xa'$, where $r \in R$ and $a' \in A$. Then $ax^k = r + x^{k+1}a'$ and $x^k(a - xa') = r \in I_k = I_{k+1}$.

Thus, we can write $x^k(a - xa') = x^{k+1}a'' + r'x$.

Dividing by x^k, we have $a - xa' = xa'' + r'x^{-k+1}$.

Thus, $a = x(a'' - a') + r'x^{-k+1}$. This means, $a \in Rx^{-k+1} + xA$, contradicting the choice of k.

Therefore, $A \subseteq Rx^{-n} + xA$, as required. □

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Corollary E2. Let R be a Noetherian domain with global transform T. Then any ring $R \subseteq A \subseteq T$ is Noetherian.

Proof. Let $I \subseteq A$ be a non-zero ideal. Take $0 \neq x \in I \cap R$. Then A/xA is Noetherian, so I/xA is finitely generated. Thus, I is finitely generated. □

Corollary F2 (Krull-Akizuki.) Let R be a one-dimensional Noetherian domain with quotient field K. Let L be a finite field extension of K. Then any ring $R \subseteq A \subseteq L$ is Noetherian. Moreover, for any prime ideal (necessarily a maximal ideal) $Q \subseteq A$ and $P = Q \cap R$, $[A/Q : R/P] < \infty$.

Proof. Since L is finite over K, there exists a finite R-module $R \subseteq R_0 \subseteq A$ such that R_0 has quotient field L.

Hence R_0 is a one-dimensional Noetherian domain and A is contained in the global transform of R_0, so A is Noetherian, by the previous corollary.

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For the second statement, take a non-zero element $x \in P$. Then A/xA is a finite A_0-module. It follows that A/Q is finite over A_0 and hence finite over A_0/Q_0, for $Q_0 := Q \cap A_0$.

Since A_0 is finite over R, A_0/Q_0 is finite over R/P.

Since $[A/Q : R/P] = [A/Q : A_0/Q_0] \cdot [A_0 : R/P]$, the proof is complete. □

The following corollary of the Krull-Akizuki theorem is very useful and plays a central role in the theory of the integral closure of ideals in Noetherian rings.

Corollary G2. Let R be a Noetherian domain with quotient field K. Given a non-zero prime ideal $P \subseteq R$, there exists a DVR $(V, \mathfrak{m}_V)$ with $R \subseteq V \subseteq K$ and $\mathfrak{m}_V \cap R = P$.

Proof. Without loss of generality, we may localize at P and assume it is the unique maximal ideal of R. Suppose $P = (a_1, \dots, a_n)R$.

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Set $T := R[\frac{a_2}{a_1}, \dots, \frac{a_n}{a_1}]$ and note $PT = a_1 T$. Let $Q \subseteq T$ be a height one prime containing $a_1 T$. Then $Q \cap R = P$.

T_Q is a one-dimensional local domain, so by the Krull-Akizuki theorem, T'_Q is Noetherian. Take a maximal ideal $\mathfrak{m} \subseteq T'_Q$ lying over QT_Q.

Then $V = (T'_Q)_{\mathfrak{m}}$ is a DVR and its maximal ideal $\mathfrak{m}_V$ has the property that $\mathfrak{m}_V \cap T = Q$.

Thus, $\mathfrak{m}_V \cap R = P$, as required. $\square$

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Remark. Corollary E2 holds if R is just a reduced Noetherian ring with total quotient ring K. Take $R \subseteq A \subseteq T$.

Since K is a localization of R, it is not difficult to see that A has finitely many minimal primes, say $Q_1, \ldots, Q_r$ and they are all of the form $QK \cap A$, for Q a minimal prime of R.

Let $Q_i \subseteq A$ be a minimal prime. Then $Q_i \cap R$ is a minimal prime.

The ring A/Q_i lies between $R/(Q_i \cap R)$ and its global transform, so A/Q_i is Noetherian.

It is also a Noetherian A-module, since the action of A on A/Q_i is the same as the action of A/Q_i on itself.

Since $A \hookrightarrow A/Q_i \oplus \cdots \oplus A/Q_r$, it follows that A is a Noetherian A-module, and hence a Noetherian ring.

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We want to present one more application of Matijevic's theorem that applies to ideal transforms - even though it is not related to the Mori-Nagata theorem.

This result shows that for local rings $(R, \mathfrak{m})$ with well behaved completions, the transform $T(\mathfrak{m})$ is a finite R-module.

In some sense, this is not saying too much, because if R has depth greater than one, $T(\mathfrak{m}) = R$. For this result we need the lemma below and the following standard fact we leave as an exercise.

Standard Fact. If $(R, \mathfrak{m})$ is a complete local ring, then R is complete in the I-adic topology, for any ideal $I \subseteq R$.

We note that the condition on the completion of R in Theorem I2 will always hold for a local ring from algebraic geometry that is reduced and equidimensional, e.g., an integral domain.

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Lemma H2. Let $(R, \mathfrak{m})$ be a local ring. Then $T(\mathfrak{m})$ is a finite R-module if and only if $T(\mathfrak{m}\widehat{R})$ is a finite $\widehat{R}$-module.

Proof. We first note that for any ideal I having grade at least one in the Noetherian ring S, if we write $I = (x_1, \ldots, x_r)S$, with each x_i a non-zerodivisor, then $T(I) = S_{x_1} \cap \cdots \cap S_{x_r}$.

Indeed, if $u \in T(I)$, then for each $1 \leq i \leq r$, $x_i^{n_i} \cdot u \in S$, for some n_i, so $u \in S_{x_i}$. Conversely, if $u \in S_{x_1} \cap \cdots \cap S_{x_r}$, then for n sufficiently large, $x_i^n \cdot u \in S$, for all i. But then $I^c \cdot u \subseteq S$, for $c >> n$., which shows $T(I) = S_{x_1} \cap \cdots \cap S_{x_r}$.

Now suppose S above is our local ring R and $I = \mathfrak{m}$. Then

$$T(\mathfrak{m}) = R_{x_1} \cap \cdots \cap R_{x_r}.$$

Therefore,

$$T(\mathfrak{m}) \otimes \widehat{R} = R_{x_1} \cap \cdots \cap R_{x_r} \otimes \widehat{R} = \widehat{R_{x_1}} \cap \cdots \cap \widehat{R_{x_r}} = T(\mathfrak{m}\widehat{R}),$$

i.e., $T(\mathfrak{m}\widehat{R}) = T(\mathfrak{m}) \otimes \widehat{R}$. By faithful flatness, $T(\mathfrak{m})$ is finite over R if and only if $T(\mathfrak{m}\widehat{R})$ is finite over $\widehat{R}$. □

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Theorem I2. Let $(R, \mathfrak{m})$ be a local ring with positive depth. Then $T(\mathfrak{m})$ is a finite R-module if and only if there does not exist $z \in \mathrm{Ass}(\widehat{R})$ with $\dim(\widehat{R}/z) = 1$.

Proof. By the previous lemma, we may assume R is complete. Suppose T is not finite over R and take $x \in R$ be a non-zerodivisor. By Matijevic's theorem, T/xT is finite over R. Thus, it must be the case that $\bigcap_{n\geq 1} x^nT \neq 0$, since R is complete in the x-adic topology.

Let $a \in R$ be a not-zero element in $\bigcap_{n\geq 1} x^nT$. Then $\frac{a}{x^n} \in T$, for all $n \geq 1$. Thus, for each $n \geq 1$, there exists $s(n) \geq 1$ with $\mathfrak{m}^{s(n)} \cdot \frac{a}{x^n} \subseteq R$.

In other words, $\mathfrak{m}^{s(n)} \subseteq (x^nR : a)$, for all n. However, there exists $k \geq 1$ such that $(x^n : a) \subseteq (0 : a) + x^{n-k}R$, for $n \geq k$ (a consequence of Artin-Rees). If $z \in \mathrm{Ass}(R)$ contains $(0 : a)$, and $n = k + 1$, we have $\mathfrak{m}^{s(n)} \subseteq x + zR$.

Thus, $\mathfrak{m}$ is minimal over $x + zR$, so $\dim(R/z) = 1$, by Krull's principal ideal theorem.

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Conversely, suppose $\dim(R/z) = 1$, for $(0:a) = z \in \mathrm{Ass}(R)$. Take $x \in R$ a non-zerodivisor.

Then for all $n \geq 1$, there exists $s(n)$ such that $m^{s(n)} \subseteq x^n R + (0:a)$. Therefore, $\mathfrak{m}^{s(n)} a \subseteq x^n$, for all n, and thus $\frac{a}{x^n} \in T(\mathfrak{m})$, for all $n \geq 1$.

But then $R \cdot \frac{a}{x} \subsetneq R \cdot \frac{a}{x^2} \subsetneq \cdots$ is a strictly increasing chain of submodules of $T(\mathfrak{m})$.

Therefore, $T(\mathfrak{m})$ is not finite over R. □

Remark. A celebrated example from the 1970s due to Ferrand and Raynaud shows that there exists a two-dimensional local domain whose completion has an associated prime z satisfying $\dim(\widehat{R}/z) = 1$. Thus, in this case, $T(\mathfrak{m})$ is not a finite R-module.

The following corollary is for those who are familiar with local cohomology.

Maijevic's Theorem

Corollary J2. Let $(R, \mathfrak{m})$ be a local ring having depth one, so that $H^1_{\mathfrak{m}}(R) \neq 0$. Then $H^1_{\mathfrak{m}}(R)$ is finite if and only if if there does not exist $z \in \mathrm{Ass}(\widehat{R})$ with $\dim(\widehat{R}/z) = 1$.

Proof. Set K to be the total quotient ring of R. Note that $\mathfrak{m}K = K$, so $H^i_{\mathfrak{m}}(K) = 0$ for all i. The short exact sequence $0 \to R \to K \to K/R \to 0$ gives rise to the long exact sequence in cohomology

$$\cdots \to H^0_{\mathfrak{m}}(K) \to H^0_{\mathfrak{m}}(K/R) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(K) \to \cdots,$$

Since $H^0_{\mathfrak{m}}(K) = H^1_{\mathfrak{m}}(K) = 0$, we have that $H^0_{\mathfrak{m}}(K/R)$ is isomorphic to $H^1_{\mathfrak{m}}(R)$.

But $H^0_{\mathfrak{m}}(K/R)$ is just $T(\mathfrak{m})/R$. Thus, $H^1_{\mathfrak{m}}(R)$ is a finite R-module if and only if $T(\mathfrak{m})/R$ is a finite R-module. But this latter module is finite over R if and only if $T(\mathfrak{m})$ is finite over R.

Thus, Theorem I2 implies that $H^1_{\mathfrak{m}}(R)$ is a finite R-module if and only if there does not exist $z \in \mathrm{Ass}(\widehat{R})$ with $\dim(\widehat{R}/z) = 1$. □